Homework 6

Linear Algebra 2
March 24, 2022 Abhiruk Lahiri

Instructions Mention your name and *code name* on top of your solution sheet. Deadline for submitting this homework is 31st March 2022, 14:00 hrs. The problem 4 carries 4 points.

Problem 1. *Let* $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. *Diagonalise* A *and compute* $\sqrt{A}$.

Problem 2. *Construct a matrix* A *whose eigenvalues are* 0 *and* 3 *and corresponding eigenvectors are* $(-1-i, 2)^T$ *and* $(1+i, 1)^T$

Problem 3. *In a linear algebra class* 50% *students prefer to solve numerical problems,* 30% *students prefer to solve coding exercises and* 20% *students prefer to do theoretical proofs. After each class, of those students who preferred to solve numerical problems,* 40% *continued to prefer solve numerical problems,* 10% *now preferred to solve coding exercises, and* 50% *preferred to do write proofs; of those who preferred to solve coding exercises,* 20% *now preferred to solve numerical problems,* 70% *continued to prefer writing codes, and* 10% *now preferred to write proofs; of those who preferred to do theoretical proofs,* 20% *now preferred to do numericals,* 20% *now preferred to do coding, and* 60% *continued to prefer theoretical proofs.*

(i) What is the percentage of students with different preferences after the first class.

(ii) What will be the distribution of students with different preferences at the end of the course assuming the course has infinitely many classes.

Hint: The situation can be described in a transition matrix

$$A = \begin{pmatrix} 0.40 & 0.20 & 0.20 \\ 0.10 & 0.70 & 0.20 \\ 0.50 & 0.10 & 0.60 \end{pmatrix}$$

You know the initial distribution P. You have been asked to compute AP and convergence of $A^m P$ when $m \to \infty$. Since A is regular the long range prediction can be found by computing the fixed probability vector for A that corresponds to the eigenvalue 1.

Notice that columns of the matrix sum to 1. *Such a matrix is called* transition matrix *or* stochastic matrix. *You will learn more about them later in your course.*

Problem 4. *Suppose that* G *is a finite graph in which any two vertices have precisely one common neighbour. Then there is a vertex which is adjacent to all other vertices.*

Hints: We prove this by contradiction. So we assume that there exists no vertex of G that is adjacent to all other vertices. One can prove such a graph has to be regular (degree of each vertex is same). We skip this part of the proof as it is beyond the scope of our course. Assume the graph is k regular. Hence total number of vertices of G is $n = k^2 - k + 1$. This clearly implies $k > 2$. For $k = 1$ we have $n = 1$ and $k = 2$ implies $n = 3$. So we have K_1 and K_3 (complete graphs of 1 and 3 vertices), both satisfy the statement.

(i) Compute the adjacency matrix A of G. Compute A^2. Notice that A^2 is a symmetric matrix and can be written as $\alpha I + J$ where I is the identity matrix and J is the all one matrix.

(ii) Compute the eigenvalues of A^2 from eigenvalues of I and J. What is the multiplicity of the eigenvalues?

(iii) What can you say about the eigenvalues of A? Argue, why they are the eigenvalues of A.

(iv) Remaining is to compute the multiplicity of the eigenvalues of A. Here we will use the fact, sum of the eigenvalues is same as the trace of the matrix.

Let the multiplicity of the two eigenvalues be r and s. Assume $h = \sqrt{k-1}$. Then the above equation tells you h divides $h^2 + 1$ and h^2. This is only possible when h equals to 1. That implies $k = 2$ which we already excluded. So we arrived at a contradiction. Hence proved. Hola! you just solved one of the many beautiful results by Paul Erdös, Alfred Rényi and Vera Sós. If you never heard of their names before, search on Google.