Linear Algebra 2
4^{th} quiz
17^{th} March, 2022

Name: .... **Central Personal ID:** ....

1. (a) Let $A, B \in \mathbb{R}^{n\times n}$ be two matrices then prove or disprove the following statement: AB and BA have the same eigenvalues.
   (b) Let $A \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that A^T has the same set of eigenvalues.

Solution to part (a). Let λ be an eigenvalue of AB. Our goal is to show that λ is also an eigenvalue of BA. As, λ is an eigenvalue of AB there exists an eigenvector v such that $ABv = \lambda v$.

$$(BA)Bv = B(AB)v = B(\lambda v) = \lambda Bv$$

If $Bv \neq 0$, then the above equation shows that λ is an eigenvalue of BA. $Bv = 0$ implies $ABv = A0 = 0$, i.e., $\lambda = 0$. $Bv = 0$ implies B is singular. B is singular implies BA is singular, which implies $\lambda(= 0)$ is an eigenvalue of BA. □

Solution to part (b). Eigenvalues of a matrix are the roots of its characteristic polynomial $p_A(\lambda) = \det(A - \lambda I_n)$. If you remember in the first lectured we learned that $\det(A) = \det(A^T)$. It is easy to observe that $\det((A - \lambda I_n)^T) = \det(A^T - \lambda I_n)$. Hence we have

$$p_{A^T}(\lambda) = \det(A^T - \lambda I_n) = \det((A - \lambda I_n)^T) = \det(A - \lambda I_n) = p_A(\lambda)$$

Hence, roots of both the polynomials are same. □

Remark 1. *Above argument does not guarantee anything about the eigenvectors of A^T in terms of the eigenvectors of A. They can differ. Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Eigenvectors of A are of the form $(1, 0)^T$ and eigenvectors of A^T are of the form $(0, 1)^T$*