Homework 3 Solution

Linear Algebra 2
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Problem 1. *A matrix is called symmetric if $A^T = A$. Prove that every symmetric 2×2 matrix has real eigenvalues.*

Solution. Let $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ be the symmetric matrix. Then the characteristic polynomial is $p(x) = x^2 - (a+b)x + (ab - h^2)$. Discriminant of $p(x)$ is $(a+b)^2 - 4(ab-h^2) = (a-b)^2 + h^2 \geq 0$. Thus the symmetric matrix or order 2 has real eigenvalues. ■

Problem 2. *Find the characteristic polynomial and eigenvalues of the following matrix*

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

Solution. The characteristic polynomial of A is

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

Expanding, we get

$$\begin{aligned} \lambda \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & 2-\lambda \\ 2 & 3-\lambda \end{vmatrix} &= 0 \\ -\lambda(2-\lambda)(3-\lambda) + 2[-2(2-\lambda)] &= 0 \\ (2-\lambda)[\lambda(\lambda-3) - 4] &= 0 \\ (2-\lambda)(\lambda-4)(\lambda+1) &= 0 \end{aligned}$$

Hence, eigenvalues are $2, -1, 4$. ■

Problem 3. *The following matrix has eigenvalues 24 and −48. What are the other two eigenvalues?*

$$A = \begin{pmatrix} 51 & -12 & -21 & 0 \\ 60 & -40 & -28 & 0 \\ 57 & -68 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

Solution. It is clear that $\det(A-\lambda I)=0$ has a factor $7-\lambda$ if we expand the determinant along the last row. Hence 7 is an eigenvalue. The other two eigenvalues are given 24 and -48. We know the sum of all eigenvalues $\sum \lambda_i = \text{trace}(A) = 51-40+1+7=25$. Hence the fourth eigenvalue is 36. ■

Problem 4. *The following matrix has eigenvalues* 3 *and* -4. *What are the other two eigenvalues?*

$$A = \begin{pmatrix} 10 & 0 & 7 & -7 \\ 4 & 5 & 2 & -2 \\ 16 & 4 & 15 & -8 \\ 30 & 4 & 26 & -19 \end{pmatrix}$$

Solution. We will use the two formula $\sum \lambda_i = \text{trace}(A)$ and $\prod \lambda_i = \det(A)$. We can compute the using Gaussian elimination method (or any other technique) $\det(A) = -420$.

$$\lambda_1 \lambda_2 3(-4) = -420$$
$$\lambda_1 \lambda_2 = 35$$

Again $\lambda_1 + \lambda_2 + 3 + (-4) = 10 + 5 + 15 - 19 = 11$. Hence $\lambda_1 + \lambda_2 = 12$.

Solving the above two equation we get $\lambda_1 = 5$ and $\lambda_2 = 7$. ■

Problem 5. *Prove or disprove the following statement: Let* A, B *be two square matrices and* $\{\alpha_i\}$ *and* $\{\beta_j\}$ *be the corresponding eigenvalues. Let eigenvalues of* AB *be* $\{\gamma_k\}$. *Then* $\sum\limits_k \gamma_k = (\sum\limits_i \alpha_i)(\sum\limits_j \beta_j)$.

Solution. Consider $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = B$. Then sum of the eigenvalues of A, B is 0. $AB = BA = A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and the sum of the eigenvalues is -2. Hence the statement is false. ■