Homework 4 Solution

Linear Algebra 2
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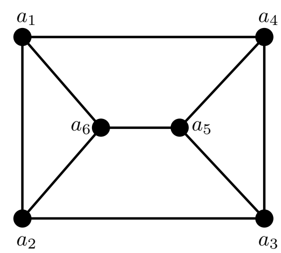
Problem 1. *Compute the number of spanning trees of the following graph*



Solution. The Lagrangian matrix of the graph is

$$L(G) = \begin{pmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & -1 & 3 \end{pmatrix}$$

We know the number of spanning trees is $= \det(L^{1,1}(G))$.

$$\det \begin{pmatrix} 3 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 3 \end{pmatrix} = 75$$

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Problem 2. *(i) Find the characteristic polynomial and compute the eigenvalues and the corresponding eigenvectors for the following matrix*

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$$

(ii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined as $T(x, y, z) = (2y, 0, 5z)$. Compute the matrix of the linear map and find all eigenvalues and eigenvectors of T.

Solution. (i) Compute the characteristic polynomial of the matrix A

$$\begin{aligned} p_A(\lambda) &= \begin{vmatrix} 2-\lambda & -1 & 2 \\ 5 & -3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} \\ &= (2-\lambda)(-3-\lambda)(-2-\lambda) + (-1)\cdot 3 \cdot (-1) - 2 \cdot (-3-\lambda)\cdot(-1) - (-1)\cdot 5 \cdot (-2-\lambda) \\ &= -(\lambda+1)^3 \end{aligned}$$

Hence A has eigenvalue $\lambda = -1$. To compute the corresponding eigenvectors, we solve the following system of equations

$$\begin{pmatrix} 2-(-1) & -1 & 2 \\ 5 & -3-(-1) & 3 \\ -1 & 0 & -2-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

That implies $3x - y + 2z = 0$, $5x - 2y + 3z = 0$ and $x + z = 0$. Hence $x = -z$ and $x = y$. So, the corresponding set of eigenvectors is $\{\alpha(1,1,-1)^T : \alpha \in \mathbb{R}\}$.

(ii) Let λ is an eigenvalue of T. Then, we can write $T(x,y,z) = \lambda(x,y,z)$ which yields the system of equations $2y = \lambda x$, $0 = \lambda y$ and $5z = \lambda z$. If $\lambda \neq 0$ then the first two equations implies $y = 0$ and $x = 0$. We know corresponding to an eigenvalue there exists a non-zero eigenvector. So the only non-zero eigenvalue is $\lambda = 5$. The set of eigenvectors corresponding to $\lambda = 5$ is $\{\alpha(0,0,1) : \alpha \in \mathbb{R}\}$.

When $\lambda = 0$ we have $y = 0$ and $z = 0$. With this values of y, z the above set of equations are satisfied for all values of x. The set of eigenvectors corresponding to the eigenvalue $\lambda = 0$ is $\{\alpha(1,0,0) : \alpha \in \mathbb{R}\}$. ■

Problem 3. *(i) Let A be an invertible matrix and $\lambda \in \mathbb{R} \setminus \{0\}$ be a number. Prove that λ is an eigenvalue of A if and only if $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}*

(ii) Let $A, D \in \mathbb{R}^{n\times n}$ and D be invertible. Prove that A and $D^{-1}AD$ have the same eigenvalues.

Solution. (i) Let λ is an eigenvalue of A. Hence there exists a non-zero eigenvector v such that $Av = \lambda v$. We know A is invertible. Multiply both sides with A^{-1} we obtain $v = \lambda A^{-1}v$, in other words $A^{-1}v = \frac{1}{\lambda}v$.

Conversely, take $A^{-1}v = \frac{1}{\lambda}v$ and multiply both sides by A to obtain the result.

(ii) Let λ is an eigenvalue of A the there exists an eigenvector $v \neq 0$ such that $Av = \lambda v$. As D is invertible there exists $u \neq 0$ such that $Du = v$. So, we have $D^{-1}ADu = D^{-1}A(Du) = D^{-1}Av = D^{-1}\lambda v = \lambda u$. Thus λ is also an eigenvalue of $D^{-1}AD$.

Conversely, if λ is an eigenvalue of $D^{-1}AD$ then there exists $v \neq 0$ such that $D^{-1}ADv = \lambda v$. Hence $D^{-1}A(Dv) = \lambda D^{-1}(Dv)$ as D is invertible and so is D^{-1}. Thus $A(Dv) = \lambda D(v)$. ■

Problem 4. *Let A be a square matrix and column rank of A be k. Prove that A has at most $k+1$ distinct eigenvalues.*

Solution. Let $\lambda_1, \lambda_2, \ldots \lambda_m$ be distinct eigenvalues of A and $v_1, v_2, \ldots, v_m$ be corresponding non-zero eigenvectors. At most one of $\lambda_1, \lambda_2, \ldots \lambda_m$ equals to 0. If $\lambda_j \neq 0$ then $A(v_j/\lambda_j) = v_j$. Hence at least $m-1$ of the vectors $v_1, v_2, \ldots, v_m$ are linearly independent and span the column space of A. Hence $m - 1 \leq \dim A = k$ implies $m \leq k + 1$. ■

Problem 5. *(i) Let $A, D \in \mathbb{R}^{n\times n}$ and D be invertible. Prove that if $p(x)$ be any polynomial (not necessary characteristic polynomial), then $p(DAD^{-1}) = Dp(A)D^{-1}$.*

(ii) Prove that α is an eigenvalue of $p(A)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of A.

Solution. (i) For any positive integer t

$$(DAD^{-1})^t = (DAD^{-1})\ldots(DAD^{-1}) = DT(D^{-1}D)A\ldots(D^{-1}D)AD^{-1} = DA^tD^{-1}$$

For any polynomial $p(x) = \sum_i a_i x^i$ if we set $x = DAD^{-1}$, we obtain $Dp(A)D^{-1}$.

(ii) Let α is an eigenvalue of $p(A)$. Thus $p(A) - \alpha I$ vanishes at multiple values. Consider the polynomial $p(x) - \alpha$. We know we can factorise a polynomial. Consider the polynomial in its factorised form $p(x) - \alpha = c(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_m)$ where $c(\neq 0), \lambda_1, \lambda_2, \ldots \lambda_m \in \mathbb{C}$. Now $p(A) - \alpha I = c(A - \lambda_1)(A - \lambda_2)\ldots(A - \lambda_m)$ vanishes at some point implies $A - \lambda_j I$ vanishes for some j. Hence λ_j is an eigenvalue of A. Also, it implies $p(\lambda_j) - \alpha = 0$. Hence $\alpha = p(\lambda_j)$.

Conversely, let $\alpha = p(\lambda)$ for some eigenvalue λ of A. Thus, there exists an eigenvector $v \neq 0$ such that $Av = \lambda v$. It is easy to observe that $A^t v = \lambda^t v$ for every positive integer t. Thus $p(A)v = p(\lambda)v = \alpha v$. Hence α is an eigenvalue of $p(A)$. ■